PERTH MODERN SCHOOL

Exceptional schooling. Exceptional students.

INDEPENDENT PUBLIC SCHOOL

WAEP Semester One Examination, 2019

Question/Answer booklet

MATHEMATICS SPECIALIST UNIT 3

Section One: Calculator-free

SOLUTIONS

Student number: In figures

In words ______________________________________

Your name ______________________________________

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answer to the specific question asked and to follow any instructions that are specified to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section One: Calculator-free **35% (52 Marks)**

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(4 marks)**

The equations of three planes are shown below.

$$\begin{aligned} x - y - z &= 3 \\ 2x + 2y + z &= 8 \\ x + y - z &= 1 \end{aligned}$$

(a) Determine the coordinates of the point of intersection of the planes. (3 marks)

Solution
$(2) - 2(3): 3z = 6 \Rightarrow z = 2$ $(1) - (3): -2y = 2 \Rightarrow y = -1$ $(3): x - 1 - 2 = 1 \Rightarrow x = 4$ Intersect at $(4, -1, 2)$
Specific behaviours
✓ solves correctly to find the first variable ✓ solves correctly to find the second and third variables ✓ answers using coordinates

(b) Determine the distance of the point of intersection of the planes from the origin. (1 mark)

Solution
$d = \sqrt{4^2 + 1^2 + 2^2} = \sqrt{21}$
Specific behaviours
✓ correct distance

DO NOT WRITE IN THIS AREA AS IT WILL BE CUT OFF

Question 2 **(6 marks)**

(a) Determine the modulus and argument of $\dfrac{1}{1+i}$. (3 marks)

Solution
$z = \dfrac{1}{1+i} \times \dfrac{1-i}{1-i} = \dfrac{1}{2} - \dfrac{i}{2}$ $\lvert z \rvert = \sqrt{\dfrac{1}{4} + \dfrac{1}{4}} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$ $\arg z = -\dfrac{\pi}{4}$
Specific behaviours
✓ real and imaginary parts   ✓ modulus   ✓ argument

(b) Determine z^3 in the form $a + bi$, where $a, b \in \mathbb{R}$, when $z = 2\cos\left(\frac{\pi}{18}\right) + 2i\sin\left(\frac{\pi}{18}\right)$.
(3 marks)

Solution
$z^3 = 8\,\mathrm{cis}\left(\dfrac{\pi}{6}\right)$ $= 8\left(\dfrac{\sqrt{3}}{2} + i\,\dfrac{1}{2}\right)$ $= 4\sqrt{3} + 4i$
Specific behaviours
✓ modulus of z^3 ✓ argument of z^3 ✓ correct rectangular form

DO NOT WRITE IN THIS AREA AS IT WILL BE CUT OFF

Question 3 **(6 marks)**

Functions f and g are defined over their natural domains by $f(x) = \sqrt{8 - x}$ and $g(x) = 3 + \dfrac{4}{\sqrt{x}}$.

(a) State the domain of

(i) $g(x)$. (1 mark)

Solution
$D_f = \{x: x \in \mathbb{R}, x > 0\}$
Specific behaviours
✓ states that $x > 0$

(ii) $g^{-1}(x)$. (2 marks)

Solution
$D_{g^{-1}} = R_g = \{x: x \in \mathbb{R}, x > 3\}$
Specific behaviours
✓ indicates $D_{g^{-1}} = R_g$ ✓ states that $x > 3$

(b) Determine $f \circ g(x)$ and the natural domain of this composite function. (3 marks)

Solution
$f \circ g(x) = \sqrt{5 - \dfrac{4}{\sqrt{x}}}$ $5 - \dfrac{4}{\sqrt{x}} \geq 0 \Rightarrow \dfrac{4}{\sqrt{x}} \leq 5$ $\therefore \sqrt{x} \geq \dfrac{4}{5}$ (since $\sqrt{x} \geq 0$) $D_{f \circ g} = \left\{x: x \in \mathbb{R}, x \geq \dfrac{16}{25}\right\}$
Specific behaviours
✓ composite function ✓ states that radicand ≥ 0 ✓ states that $x \geq \frac{16}{25}$

DO NOT WRITE IN THIS AREA AS IT WILL BE CUT OFF

Question 4 **(6 marks)**

(a) State whether the planes with equations $2x - y + z = 2$ and $x + 3y + 2z = 1$ are perpendicular. Justify your answer. (2 marks)

Solution
$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = 2 - 3 + 2 = 1$ Planes not perpendicular as normals are not perpendicular.
Specific behaviours
✓ calculates dot product of normals   ✓ states correct conclusion

(b) Determine the Cartesian equation of the plane that passes through the three points with position vectors shown below. (4 marks)

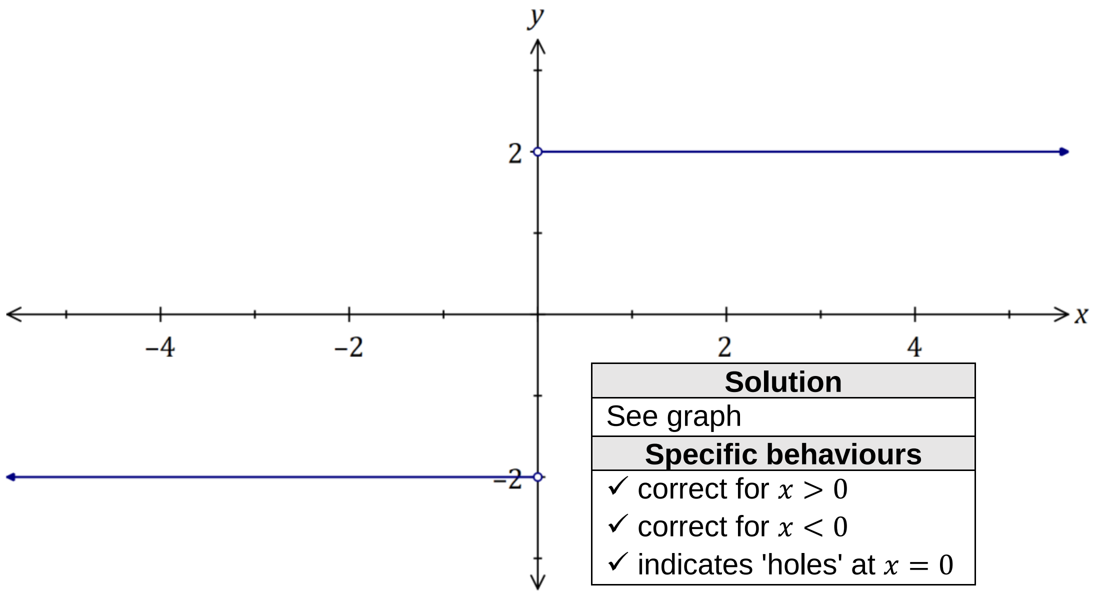
$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

Solution
$AC = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, BC = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \\ 2 \end{pmatrix}$ $\begin{pmatrix} -4 \\ -3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = -10$ $4x + 3y - 2z = 10$
Specific behaviours
✓ obtains two vectors in plane   ✓ calculates cross product   ✓ uses dot product to obtain constant   ✓ states equation in correct form

DO NOT WRITE IN THIS AREA AS IT WILL BE CUT OFF

Question 5 **(7 marks)**

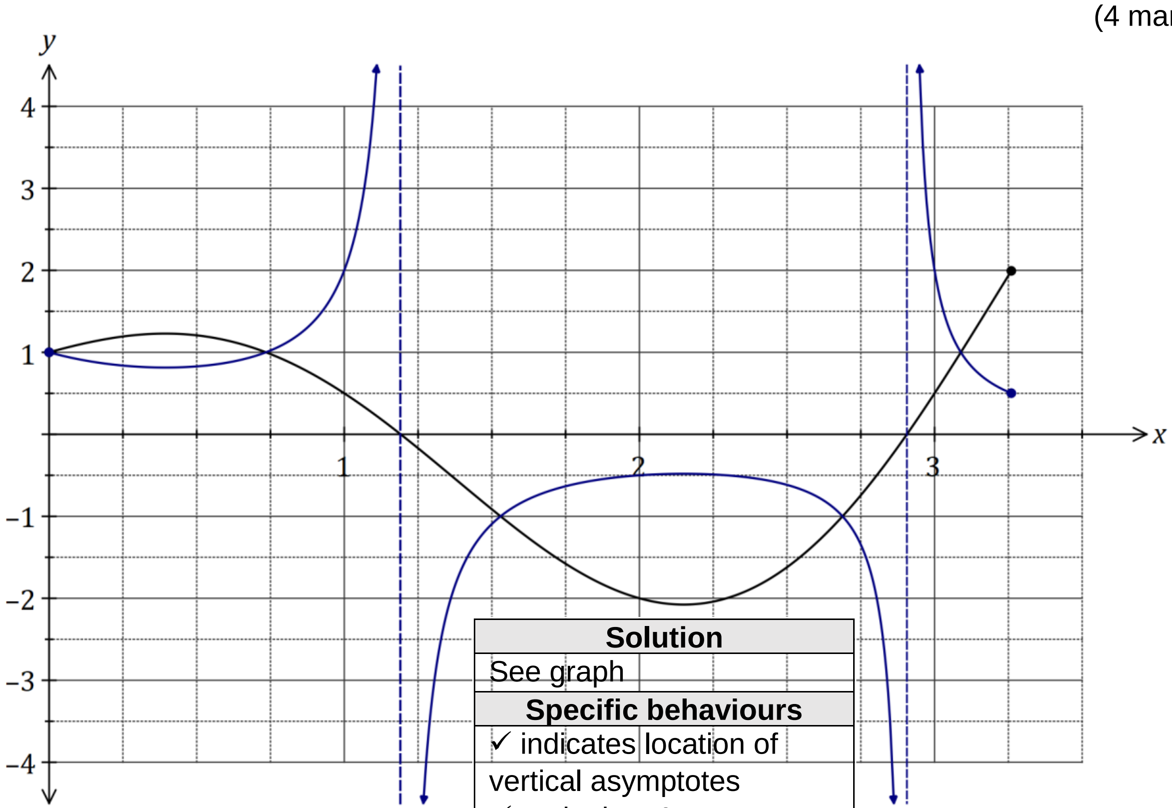
(a) On the axes below, sketch the graph of $y = \dfrac{2x}{|x|}$. (3 marks)



Solution
See graph
Specific behaviours
✓ correct for $x > 0$
✓ correct for $x < 0$
✓ indicates 'holes' at $x = 0$

(b) The graph of $y = f(x)$ is shown below. On the same axes draw the graph of $y = \dfrac{1}{f(x)}$.

(4 marks)



Solution
See graph
Specific behaviours
✓ indicates location of vertical asymptotes
✓ endpoints & intersections at $y = \pm 1$
✓ correct behaviour & max between asymptotes
✓ correct behaviour & min outside asymptotes

Question 6 **(7 marks)**

Four functions are defined as

$$f(x) = 2x^2 - x - 1, \qquad g(x) = x^2 - 2x - 3, \qquad h(x) = x + 1, \qquad k(x) = x - 3$$

Determine the equations of all asymptotes of the following graphs.

(a) $y = \dfrac{h(x)}{g(x)}$. (2 marks)

Solution
$y = \dfrac{x+1}{(x-3)(x+1)} = \dfrac{1}{x-3}, x \neq -1.\ x \to \infty, y \to 0$ Asymptotes: $x = 3,\ \ y = 0$
Specific behaviours
✓ vertical asymptote   ✓ horizontal asymptote

(b) $y = \dfrac{g(x)}{f(x)}$. (2 marks)

Solution
$y = \dfrac{(x-3)(x+1)}{(2x+1)(x-1)}, \qquad x \to \infty, y \to \dfrac{1}{2}$ Asymptotes: $x = -\frac{1}{2},\ x = 1,\ y = \frac{1}{2}$
Specific behaviours
✓ vertical asymptotes   ✓ horizontal asymptote

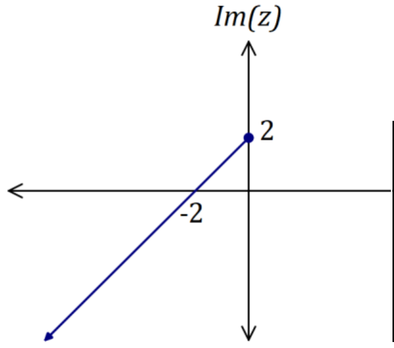
(c) $y = \dfrac{f(x)}{k(x)}$. (3 marks)

Solution
$y = \dfrac{2x^2 - x - 1}{x - 3} = \dfrac{2x^2 - 6x}{x - 3} + \dfrac{5x - 15}{x - 3} + \dfrac{14}{x - 3}$ $= 2x + 5 + \dfrac{14}{x - 3}.\ x \to \infty, y \to 2x + 5$ Asymptotes: $x = 3,\ y = 2x + 5$
Specific behaviours
✓ obtains quotient   ✓ vertical asymptote   ✓ oblique asymptote

DO NOT WRITE IN THIS AREA AS IT WILL BE CUT OFF

Question 7 **(8 marks)**

(a) Sketch the locus of points z in the complex number determined by $\arg(z - 2i) = -\frac{3\pi}{4}$. (3 marks)



Solution
See diagram
Specific behaviours
✓ ray ✓ ray starts at $2i$ ✓ axes-intercepts or shows argument

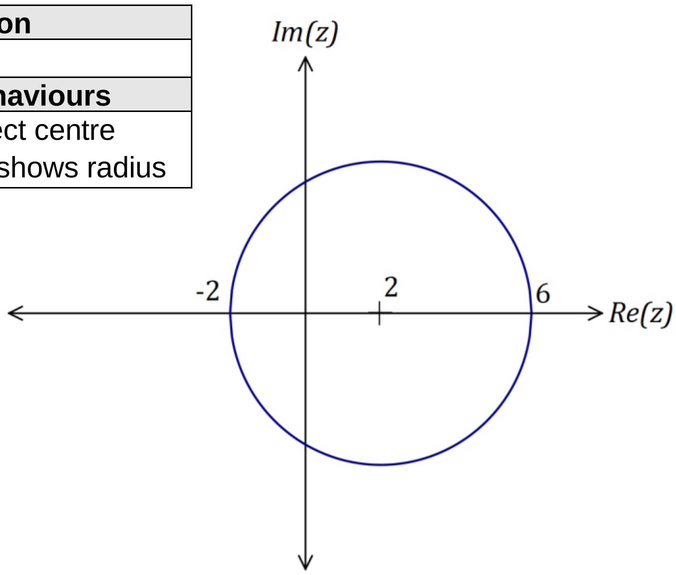
(b) Another locus of points z in the complex plane is determined by $z\bar{z} - 2z - 2\bar{z} = 12$.

(i) Show that this locus can also be defined in the form $|z - w| = k$, clearly showing the value of constant w and the value of constant k. (3 marks)

Solution
Let $z = x + iy \Rightarrow z\bar{z} - 2z - 2\bar{z} = x^2 + y^2 - 2x - 2y - 2x + 2y$ $x^2 - 4x + y^2 = 12$ $(x-2)^2 + y^2 = 16$ Hence $\lvert z - 2 \rvert = 4$
Specific behaviours
✓ expands using real and imaginary parts ✓ shows circle in factored Cartesian form ✓ writes using magnitude form

(ii) Sketch the locus on the axes below. (2 marks)

Solution
See diagram
Specific behaviours
✓ circle with correct centre ✓ x-intercepts or shows radius



Question 8 **(8 marks)**

Let $z = x + yi$ and $z^2 = a + bi$ where $a, b, x, y \in \mathbb{R}$.

(a) Show that $\sqrt{a^2 + b^2} + a = 2x^2$. (4 marks)

Solution
$z^2 = x^2 - y^2 + 2xyi$ $a = x^2 - y^2, b = 2xy$ $a^2 + b^2 = x^4 - 2x^2y^2 + y^4 + 4x^2y^2$ $= x^4 + 2x^2y^2 + y^4$ $= (x^2 + y^2)^2$ $\sqrt{a^2 + b^2} + a = x^2 + y^2 + x^2 - y^2$ $= 2x^2$
Specific behaviours
✓ expression for z^2 ✓ writes expression for $a^2 + b^2$ ✓ simplifies expression for $a^2 + b^2$ ✓ expression for $\sqrt{a^2 + b^2} + a$

(b) By solving the equation $z^4 - 16z^2 + 100 = 0$ for z^2 or otherwise, determine the roots of the equation in Cartesian form. (4 marks)

Solution
$z^4 - 16z^2 + 100 = 0$ $(z^2 - 8)^2 = 64 - 100 = -36$ $z^2 = 8 \pm 6i$ $a = 8, b = \pm 6$ $2x^2 = \sqrt{8^2 + 6^2} + 8$ $x^2 = 9$ $x = \pm 3$ $y^2 = x^2 - a = 9 - 8 = 1$ $y = \pm 1$ $z = 3 + i, 3 - i, -3 + i, -3 - i$
Specific behaviours
✓ solves for z^2 ✓ solves for x^2 ✓ solves for y^2 ✓ four correct solutions

End of questions

Supplementary page

Question number: _________